STAT0041: Stochastic Calculus

Lecture 12 - Stochastic Differential Equations

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Key concepts:

- *Stochastic differential equations;*
- *Existence and uniqueness of solutions.*

12.1 Introduction

Stochastic differential equations (SDEs) are mathematical models describe random phenomena in physics, finance, and etc.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathrm{P})$ be a filtrated probability space satisfies usual conditions, B_t be a $\mathscr{F}_t$-adapted Brownian motion. We consider following Itô stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t. \tag{12.1}$$

Actually, it is supposed to be a stochastic integral equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \tag{12.2}$$

where $b(t, x)$ called *drift coefficient* and $\sigma(t, x)$ called *diffusion coefficient* are $\mathscr{F}_t$-adapted.

Example 12.1 (Ornstein–Uhlenbeck processes/Langevin equation) *Langevin proposed a possible improvement of mathematical model of the motion of a Brownian particle which models frictional forces as follows for the one-dimensional case:*

$$\dot{X} = -bX_t + \sigma\xi_t,$$

where ξ_t is "white noise", $b > 0$ is a coefficient of friction, and σ is a diffusion coefficient. We form this into SDEs

$$\begin{cases} dX_t = -bX_t dt + \sigma dB_t \\ X(0) = X_0, \end{cases} \tag{12.3}$$

for some initial distribution X_0, independent of the Brownian motion. This is the Langevin equation*. In this interpretation X_t is the velocity of the Brownian particle.*

Example 12.2 (Geometric Brownian Motion) *Let S_t denote the price of a stock at time t. We can model the evolution of S_t in time by supposing that $\frac{dS_t}{S_t}$, the relative change of price, evolves according to the SDE*

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

12.2 Existence and uniqueness of solutions

Theorem 12.3 (Existence and uniqueness of (strong) solution) *If that drift coefficient b and diffusion coefficient σ satisfy:*

(1) There exist constant C, such that

$$|b(t,x)|+|\sigma(t,x)| \leqslant C(1+|x|), \quad \forall x \in \mathbb{R}^n, t \in [0,T],$$

(2) There exist constant K, such that

$$|b(t,x)-b(t,y)|+|\sigma(t,x)-\sigma(t,y)| \leqslant K|x-y|, \quad \forall x,y \in \mathbb{R}^n, t \in [0,T].$$

Then for all $\mathscr{F}_0$-measurable random variable ξ, there exist unique $\mathscr{F}_t$-adapted X_t, satisfies

$$\begin{cases} \mathrm{d}X_t = b\left(t, X_t\right)\mathrm{d}t + \sigma\left(t, X_t\right)\mathrm{d}B_t, \\ X_0 = \xi. \end{cases}$$